QED in external field with space-time uniform invariants: Exact solutions

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(April 20, 2000)

We study exact solutions of Dirac and Klein-Gordon equations and Green functions in d-dimensional QED and in an external electromagnetic field with constant and homogeneous field invariants. The cases of even and odd dimensions are considered separately, they are essentially different. We direct attention to the asymmetry of the quasienergy spectrum, which appears in odd dimensions. The *in* and *out* classification of the exact solutions as well as the completeness and orthogonality relations is strictly proven. Different Green functions in the form of sums over the exact solutions are constructed. The Fock-Schwinger proper time integral representations of these Green functions are found. As physical applications we consider the calculations of different quantum effects related to the vacuum instability in the external field. For example, we present mean values of particles created from the vacuum, the probability of the vacuum remaining a vacuum, the effective action, and the

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expectation values of the current and energy-momentum tensor.

11.10.Kk, 04.62.+v, 12.20.Ds

I. INTRODUCTION

Exact solutions of the relativistic wave equations (Klein-Gordon or Dirac equation) in an external electromagnetic field as well as Green functions of those equations are very important in QED with such a field. Special complete sets of the exact solutions and special kinds of the Green functions allow one to calculate different quantum effects, for example, particles scattering, pair creation and so on, in zero order with respect to the radiative corrections, but taking the interaction with the external field into account exactly [1–3]. They may serve also as a basis for the perturbation theory with respect to the radiative interaction, in which the external field is taken into account exactly (so called generalized Furry picture) [4,5]. In four-dimensional space-time the above exact solutions were studied for different configurations of the external field in numerous papers and books (see, for example, [3,6]. Among all the configurations of the external electromagnetic fields, which admit the exact solutions in $d = 4$, one is especially important due to the fact that it corresponds to a wide class of real physical situations. It is a combination of constant uniform electric and magnetic fields and plane wave field. The exact solutions can be found if there exist a reference frame in which the constant electric and magnetic fields, and the wave 3-vector are collinear. First, the exact solutions of Klein-Gordon and Dirac equation in such a configuration were found in [7]. Some of the Green functions of scalar field in the same configuration were studied in the papers [8]. Using those exact solutions and Green functions different quantum effects were calculated [9]. Calculation of Green functions of spinor field was not present in the literature. Moreover, lately, field theoretical models in dimensions different from $d = 4$ attract attention due to various reasons. One can mention here models in $2 + 1$ dimensions which probably describe planar physical phenomenon, and models in $d > 4$ dimensions in connection with Kaluza-Klein ideas (see e.g. [10]. That is why in the present article we are going to study exact solutions and Green functions (both scalar and spinor case) in d-dimensional QED with an external electromagnetic field, which may be considered as a generalization of the above mentioned special configuration in four dimensions. At the same time we also present new results in four dimensions. A general definition of such a configuration valid for any dimensions of space-time is the following: an external electromagnetic field with constant and homogeneous field invariants. Also, an important motivation to consider exact solutions and Green functions in such a configuration of the external electromagnetic field is the fact that formally problems in some configurations of an external gravitational field lead to the same kind of equations, so that the former exact solutions can be used in the latter case after some simple identifications. Such a reduction may be done, for example, for De Sitter and FRW metrics [11,12], for four fermionic models in an external field and so on.

In a sense, the paper can be considered as a continuation and generalization of the one [13], where only an electric-like external field was considered. The latter paper can help the reader to better understand the logic of the exact solutions constructed in this more complicated present case. Thus, here we sometimes omit technical details and explanations which can be extracted from the above mentioned previous paper.

The paper is organized as follows: In Sec. II we describe first the external electromagnetic field under consideration. Then, we present the so called in and out complete sets of exact solutions of Dirac and Klein-Gordon equations in the external field. The cases of even and odd dimensions have to be considered separately, they are essentially different. We attract attention to the asymmetry of the quasienergy spectrum, which appears in odd dimensions. We spend enough time to prove strictly the *in* and *out* classification of the exact solutions as well as to prove the completeness and orthogonality relations. Then in Sec. III we construct different Green functions in the form of sums over the exact solutions and present them in the form of contour integrals over the Fock-Schwinger proper time. Among them are the causal Green function, $in-in$ and $out-out$ Green functions, the commutator function and so on. One ought to say that the spinor case is treated first, even in $d=4$ dimensions. As physical applications we consider in Sec. IV the calculations of different quantum effects related to the vacuum instability in the external field. For example, we present mean values of particles created from the vacuum, the probability for the vacuum remaining a vacuum, the effective action, the expectation values of the current and energy-momentum tensor. We calculate the latter quantities by means of both *in* and *out* sets of the exact solutions.

II. COMPLETE SETS OF EXACT SOLUTIONS

The Dirac equation in an external electromagnetic field with potentials $A_\mu(x)$ in d dimensions has the form ($\hbar = c = 1$)

$$\left(\mathcal{P}_\mu \gamma^\mu - m\right)\psi(x) = 0\ , \quad \mathcal{P}_\mu = i\partial_\mu - qA_\mu(x)\ , \tag{1}$$

where $\psi(x)$ is a $2^{[\frac{d}{2}]}$-component column, γ^μ are γ-matrices in d dimensions [14],

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \mathrm{diag}(\underbrace{1, -1, -1, \ldots}_{d}), \quad d = D + 1,$$

and $x = (x^\mu) = (x^0, \mathbf{x}),\ \mathbf{x} = (x^i), \quad \mu = 0, 1, \ldots, D, \quad i = 1, \ldots, D$. The time-independent scalar product of the solutions of the Dirac equation may be chosen in the conventional form

$$(\psi, \psi') = \int \bar{\psi}(x)\gamma^0\psi'(x)d\mathbf{x}. \tag{2}$$

As usual, it is convenient to present $\psi(x)$ in the form

$$\psi(x) = \left(\mathcal{P}_\mu \gamma^\mu + m\right)\phi(x)\ . \tag{3}$$

Then the functions ϕ have to obey the squared Dirac equation in d dimensions,

$$\left(\mathcal{P}^2 - m^2 - \frac{q}{2}\sigma^{\mu\nu}\mathcal{F}_{\mu\nu}\right)\phi(x) = 0\ , \quad \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\ ,\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\ . \tag{4}$$

To construct the above mentioned generalized Furry picture in QED with an external field one has to find special sets of exact solutions of Eq. (1), namely, two complete and orthonormal sets of solution: $\left\{{}_\pm\psi_{\{n\}}(x)\right\}$ which describes particles (+) and antiparticles (−)

in the initial time instant ($x^0 \to -\infty$), and $\left\{ {}^{\pm}\psi_{\{n\}}(x) \right\}$ which describes particles (+) and antiparticles (−) in the final time instant ($x^0 \to +\infty$). According to the general approach [5] such solutions obey the following asymptotic conditions

$$\begin{aligned} &H_{o.p.}(x^0)\ {}_{\zeta}\psi_{\{n\}}(x) = {}_{\zeta}\varepsilon\ {}_{\zeta}\psi_{\{n\}}(x), \quad , \mathrm{sgn}\ {}_{\zeta}\varepsilon = \zeta,\ x^0 \to -\infty\ , \\ &H_{o.p.}(x^0)\ {}^{\zeta}\psi_{\{l\}}(x) = {}^{\zeta}\varepsilon\ {}^{\zeta}\psi_{\{l\}}(x),\ \mathrm{sgn}\ {}^{\zeta}\varepsilon = \zeta,\ \ x^0 \to +\infty\ , \end{aligned} \tag{5}$$

where $\zeta, \{n\}$ and $\zeta, \{l\}$ are complete sets of quantum numbers which characterize solutions ${}_{\zeta}\psi_{\{n\}}(x)$ and ${}^{\zeta}\psi_{\{l\}}(x)$ respectively, $H_{o.p.} = \gamma^0(m - \gamma^i \mathcal{P}_i)$ is a one-particle Dirac Hamiltonian in convenient external field gauge $A_0(x) = 0$; ${}^{+}\varepsilon$, ${}_{+}\varepsilon$ are particle quasi-energies and $|{}^{-}\varepsilon|$ and $|{}_{-}\varepsilon|$ are antiparticles quasi-energies. All the information about the processes of particles scattering and creation by an external field (in zeroth order with respect to the radiative corrections) can be extracted from the decomposition coefficients (matrices) $G\left({}_{\zeta}|^{\zeta'}\right)$,

$${}^{\zeta}\psi(x) = {}_{+}\psi(x) G\left({}_{+}|^{\zeta}\right) + {}_{-}\psi(x) G\left({}_{-}|^{\zeta}\right)\ . \tag{6}$$

The matrices $G\left({}_{\zeta}|^{\zeta'}\right)$ obey the following relations,

$$\begin{aligned} &G\left({}_{\zeta}|^{+}\right) G\left({}_{\zeta}|^{+}\right)^{\dagger} + G\left({}_{\zeta}|^{-}\right) G\left({}_{\zeta}|^{-}\right)^{\dagger} = \mathbf{I}, \\ &G\left({}_{+}|^{+}\right) G\left({}_{-}|^{+}\right)^{\dagger} + G\left({}_{+}|^{-}\right) G\left({}_{-}|^{-}\right)^{\dagger} = 0\ , \end{aligned} \tag{7}$$

where $\mathbf{I}$ is the identity matrix.

Let us describe the external electromagnetic field in which we are going to construct the exact solutions. Such a field is a generalization of the corresponding field in $d = 4$, which is a combination of the constant uniform field with the plane wave field and which admits there exact solutions. One can describe such a field in $d = 4$ in arbitrary reference frame saying that both its field invariants do not depend on space-time coordinates. In $d > 4$ there exist, generally speaking, more independent field invariants. Besides $I_1 = 1/2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, those are all possible independent invariant combinations which may be constructed from the field tensor $\mathcal{F}_{\mu\nu}$ and Levi- Civita tensor. It is easy to see that there exist $[d/2]$ such invariants. We define the external field under consideration in arbitrary dimensions in the same manner, all its field invariants have to be constant and uniform. One can see that such a field is a combination of a constant uniform field $F_{\mu\nu}$ and a plane-wave field $f_{\mu\nu}(nx)$

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu}(nx), \tag{8}$$

where n_μ is an real isotropic vector, $n_\mu n^\mu = 0$. It is an eigenvector of the tensor $F_{\mu\nu}$, $F_{\mu\nu}n^\nu = \mathcal{E}n_\mu$, , and $f_{\mu\nu}(nx)$ is a transverse field with respect to n_μ, $n^\mu f_{\mu\nu}(nx) = f_{\mu\nu}(nx)n^\nu = 0$. If all of the invariants are equal to zero (it is only possible for $d > 2$), only a plane-wave field configuration is possible. If some of the invariants are not equal to zero, then a constant, uniform component $F_{\mu\nu}$, is not zero. The field (8) is free. Such a field is of special interest due to the fact that QED with a free classical field can be treated as exact QED (without external fields) but with some special (coherent) initial photon states [15,3].

If the eigenvalue $\mathcal{E}$ is not zero, the external field violates the vacuum stability (creating particles). One can find an inertial frame where the matrix $\mathcal{F}_{\mu\nu}$ has a simple form

$$\begin{aligned}
& F_{\mu\nu} = F^{\perp}_{\mu\nu} + F^{\parallel}_{\mu\nu}, \\
& F^{\parallel}_{\mu\nu} = E\left(\delta^0_\mu \delta^D_\nu - \delta^D_\mu \delta^0_\nu\right), \\
& f_{\mu\nu}\left(nx\right) = \sum_{k=1}^{D-1} \left(n_\mu \delta^k_\nu - n_\nu \delta^k_\mu\right) \dot{f}_k\left(x_-\right), \;\; \dot{f}_k(x_-) = \frac{df_k(x_-)}{dx_-}, \\
& n_\mu = \delta^0_\mu - \delta^D_\mu, \;\; x_- = nx = x^0 - x^D \,.
\end{aligned} \tag{9}$$

If $d > 2$ is even then

$$F^{\perp}_{\mu\nu} = \sum_{j=1}^{(d-2)/2} H_j(\delta^{2j}_\mu \delta^{2j-1}_\nu - \delta^{2j}_\nu \delta^{2j-1}_\mu)\,,$$

and if $d = 2$ the fields $F^{\perp}_{\mu\nu}$ and $f_{\mu\nu}\left(nx\right)$ are absent. In the case d is odd, and $\mathcal{E} \neq 0$,

$$F^{\perp}_{\mu\nu} = \sum_{j=1}^{(d-3)/2} H_j(\delta^{2j}_\mu \delta^{2j-1}_\nu - \delta^{2j}_\nu \delta^{2j-1}_\mu) \text{ if } d > 3, \text{ and } F^{\perp}_{\mu\nu} = 0 \text{ if } d = 3\,.$$

The same formula is valued if $\mathcal{E} = 0$, but at least one of the imaginary eigenvalues of $F_{\mu\nu}$ is zero. Here $f_k(x_-)$ are arbitrary functions of x_-. If $\mathcal{E} = 0$, and all imaginary eigenvalues of $F_{\mu\nu}$ are not equal to zero (all field invariants are not equal to zero) we have

$$E=0,\ F^{\perp}_{\mu\nu}=\sum_{j=1}^{(d-1)/2}H_j(\delta^{2j}_{\mu}\delta^{2j-1}_{\nu}-\delta^{2j}_{\nu}\delta^{2j-1}_{\mu}),\quad f_k\,(x_-)=0 \text{ for all } k\,.$$

In the reference frame under consideration $\mathcal{E}=E$.

To realize the external electromagnetic field of the above form we select the following potentials:

$$\begin{aligned}
&A_{\mu}\,(x)=A^{E}_{\mu}\,(x)+A^{H}_{\mu}\,(x)+f_{\mu}\,(x_-)\,, \\
&A^{E}_{\mu}\,(x)=Ex^0\delta^{D}_{\mu},\ A^{H}_{i}=-H_j\,x_{i+1}\,\delta_{i,2j-1},\ j=1,\ldots,[d/2]-1,\ i=1,\ldots,D-1, \\
&f_{\mu}\,(x_-)=0 \text{ if } \mu=0,D\,.
\end{aligned} \tag{10}$$

Below we present linearly independent sets of solutions of the squared Dirac equation, which correspond to the particles in the initial time instance and to the antiparticles in the final time instant:

$$\begin{aligned}
{}^{-}_{+}\phi_{p_-,n,\xi,r}(x) &= {}^{-}_{+}\phi_{p_-,n,r}(x^0,x^D)\ {}^{-}_{+}u_{\xi,r}(x_-)\phi_{n,r}(x_\perp), \\
\phi_{n,r}(x_\perp) &= \phi_{p_1,n_1}(x_1,x_2)\phi_{p_3,n_2}(x_3,x_4)...\phi_{p_{d-3},n_{(d-2)/2}}(x_{D-2},x_{D-1}),\ \text{ if } d \text{ is even}, \\
\phi_{n,r}(x_\perp) &= \phi_{p_1,n_1}(x_1,x_2)\phi_{p_3,n_2}(x_3,x_4)...\phi_{p_{d-2},n_{(d-3)/2}}(x_{D-3},x_{D-2}) \\
&\qquad (2\pi)^{-1/2}\exp(-ip_{D-1}x^{D-1})\ \text{ if } d \text{ is odd and } H_{(d-1)/2}=0, \\
{}^{-}_{+}\phi_{p_-,n,r}(x^0,x^D) &= (4\pi)^{-1/2}\exp\left\{\frac{i}{2}\left(qE(x_-^2/2-x_D^2)-p_-x_++\lambda\ln\left(\mp\tilde{\pi}_-\right)\right)\right. + \\
&\qquad \left. {}^{-}_{+}J(x_-)-{}^{-}_{+}K^{\mu}(x_-)\pi_{\perp\mu}\right\}, \\
{}^{-}_{+}J\,(x_-) &= -\frac{1}{2qE}\int_{\mp\infty}^{\pi_-}qf\left(\frac{p_--\tau}{qE}\right)\left[qf\left(\frac{p_--\tau}{qE}\right)+qF\,{}^{-}_{+}K\left(\frac{p_--\tau}{qE}\right)\right]\tau^{-1}d\tau, \\
{}^{-}_{+}K\,(x_-) &= -\frac{1}{qE}\int_{\mp\infty}^{\pi_-}\exp\left\{-\frac{F}{E}\ln\frac{\pi_-}{\tau}\right\}qf\left(\frac{p_--\tau}{qE}\right)\tau^{-1}d\tau, \\
{}^{-}_{+}u_{\xi,r}(x_-) &= \mp\quad\left(\frac{1}{2\pi_-}(1+\gamma^0\gamma^D)+\frac{1}{2}(1-\gamma^0\gamma^D)+\frac{1}{2\pi_-}(\gamma^0-\gamma^D)\gamma qf(x_-)\right)v_{\xi,r}, \\
&\quad x_+=x^0+x^D,\ \tilde{\pi}_-=\pi_-/\sqrt{qE},
\end{aligned} \tag{11}$$

where p_-, $n=\left(n_1,n_2,\ldots,n_{[d/2]-1};p_1,p_3,\ldots,p_{2[(d-1)/2]-1}\right)$, ξ and $r=(r_1,r_2,\ldots,r_{[d/2]-1})$ is a complete set of quantum numbers. Among them p_-, p_j are momenta of the continuous spectrum, n_j are integer quantum numbers, $\xi = \pm 1$ and $r_j = \pm 1$ are spin quantum numbers; the momentum p_- is the eigenvalue of the operator $2i\frac{\partial}{\partial x_+}$; if $qE > 0$ is chosen then the signs $\mp$ assigned to the functions ${}_{+}^{-}\phi_{p_-,n,\xi,r}(x)$ are matched with those of the kinetic momentum $\pi_- = p_- - qEx_-$ at $x_- \to \pm\infty$, and

$$
\begin{aligned}
& x_\perp^\mu = 0 \text{ if } \mu = 0, D,\ x_\perp^\mu = x^\mu \text{ if } \mu = 1, \dots, D-1, \\
& \pi_{\perp\mu} = 0 \text{ if } \mu = 0, D,\quad \pi_{\perp\mu} = i\frac{\partial}{\partial x^\mu} - qA_\mu^H(x) \text{ if } \mu = 1, \dots, D-1, \\
& qE\lambda = m^2 + \sum_{j=1}^{[d/2]-1} \omega_j + \omega_0, \quad \omega_0 = \begin{cases} 0, & d \text{ is even} \\ p_{d-2}^2, & d \text{ is odd}\ , \end{cases} \\
& \omega_j = \begin{cases} |qH_j|(2n_j + 1 - r_j), & H_j \neq 0 \\ p_{2j-1}^2 + p_{2j}^2\ , & H_j = 0 \end{cases} .
\end{aligned}
\tag{12}
$$

Each function $\phi_{p_{2j-1},n_j}(x_{2j-1}, x_{2j})$ obeys the following equations

$$
\begin{aligned}
(\pi_{\perp 2j-1}^2 + \pi_{\perp 2j}^2 - \omega_j)\phi_{p_{2j-1},n_j}(x_{2j-1}, x_{2j}) &= 0, \\
(\pi_{\perp 2j-1} - p_{2j-1})\, \phi_{p_{2j-1},n_j}(x_{2j-1}, x_{2j}) &= 0.
\end{aligned}
$$

If $H_j \neq 0$, a solution of these equations is

$$
\phi_{p_{2j-1},n_j}(x_{2j-1}, x_{2j}) =
$$
$$
\left(\frac{\sqrt{|qH_j|}}{2^{n_j+1}\pi^{\frac{3}{2}} n_j!}\right)^{1/2} \exp\left\{-ip_{2j-1}x^{2j-1} - \frac{|qH_j|}{2}\left(x^{2j} + \frac{p^{2j-1}}{qH_j}\right)^2\right\} \mathcal{H}_{n_j}\left[\sqrt{|qH_j|}\left(x^{2j} + \frac{p^{2j-1}}{qH_j}\right)\right],
$$

where $\mathcal{H}_{n_j}(x)$ are the Hermite polynomial with integer $n_j = 0, 1, \ldots$. If $H_j = 0$, the discrete quantum numbers n_j have to be replaced by the momenta p_{2j}, and the corresponding function has the form

$$
\phi_{p_{2j-1},p_{2j}}(x_{2j-1}, x_{2j}) = (2\pi)^{-1} \exp\left\{-i\left(p_{2j-1}x^{2j-1} + p_{2j}x^{2j}\right)\right\}.
$$

The symbol $\ln(\mp\tilde{\pi}_-)$ means the principal branch of the logarithm, $\ln(\mp\tilde{\pi}_-) = \ln|\tilde{\pi}_-| + i\pi\Theta(\pm\tilde{\pi}_-)$, while the integration paths in the τ-plane, as well as the arguments π_-, are shown in FIG.1 and FIG.2 for the functions ${}_{+}K(x_-)$, ${}_{+}J(x_-)$ and ${}^{-}K(x_-)$, ${}^{-}J(x_-)$ respectively.

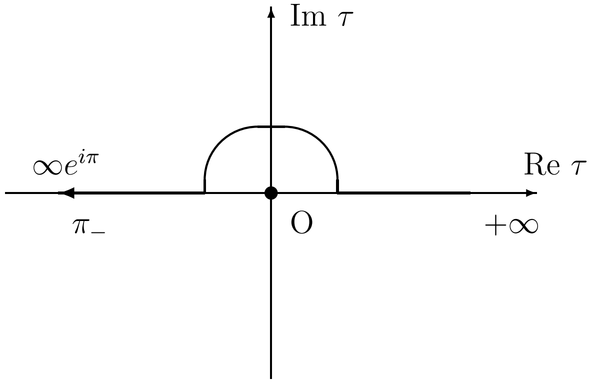

FIG. 1. Pass of integration in ${}_{+}K(x_-)$ and ${}_{+}J(x_-)$ integrals.

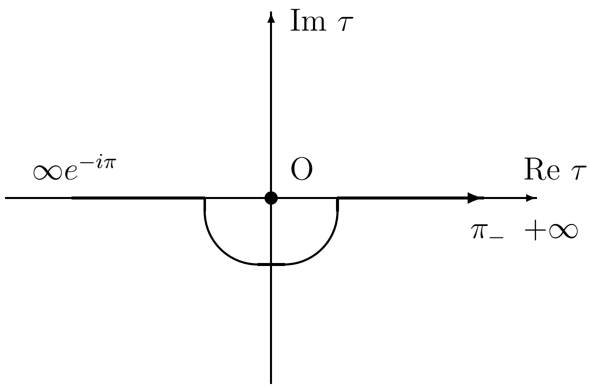

FIG. 2. Pass of integration in ${}^{-}K(x_-)$ and ${}^{-}J(x_-)$ integrals.

One assumes the functions $f_k(x_-)$ obey the requirement that ${}^{-}_{+}K\,(x_-)$ and ${}^{-}_{+}J\,(x_-)$ are analytic functions, and in particular, the functions $f_k(x_-)$ vanish at $x_- \to \pm\infty$ quite rapidly.

The solution has a different form if d is odd, $\mathcal{E}=0$, and all the imaginary eigenvalues of $F_{\mu\nu}$ are not equal to zero (in this case a plane-wave field is absent),

$$
\begin{aligned}
{}^{-}_{+}\phi_{n,\xi,r}(x) &= {}^{-}_{+}\phi_{n,r}(x^0)\phi_{n,r}(\mathbf{x})v_{\xi,r}, \\
\phi_{n,r}(\mathbf{x}) &= \phi_{p_1,n_1}(x_1,x_2)\phi_{p_3,n_2}(x_3,x_4)...\phi_{p_{d-2},n_{(d-1)/2}}(x_{D-1},x_D)\ , \\
{}^{-}_{+}\phi_{n,r}(x^0) &= c\exp\left(\pm i\left|{}_{\mp}\varepsilon_{nr}\right|x^0\right),\ \left|{}_{\mp}\varepsilon_{nr}\right| = \sqrt{m^2+\sum_{j=1}^{(d-1)/2}\omega_j},
\end{aligned}
\tag{13}
$$

where c is a normalization constant.

Here $v_{\xi,r}$ are some constant orthonormal spinors, $v^\dagger_{\xi,r}v_{\xi,r'} = \delta_{r,r'}$. Equation (4) allows one to subject these spinors to some supplementary conditions,

$$\Xi_{\pm}v_{\mp 1,r}=0,\ \Xi_{\pm}=\frac{1}{2}(1\pm\gamma^{0}\gamma^{D}),\ \mathrm{rank}\ \Xi_{\pm}=J_{(d)}=2^{[\frac{d}{2}]-1}; \tag{14}$$

$$R_{j}v_{\xi,r}=r_{j}v_{\xi,r},\ d\geq 4,\ R_{j}=i\gamma^{2j-1}\gamma^{2j}. \tag{15}$$

If $d=2,3$, the independent quantum number r does not appear.

One can verify the solutions of the Dirac equation with different ξ, namely, $(\mathcal{P}_{\mu}\gamma^{\mu}+m)\ {}_{+}\phi_{p_{-},n,+1,r}(x)$ and $(\mathcal{P}_{\mu}\gamma^{\mu}+m)\ {}_{+}\phi_{p_{-},n,-1,r}(x)$, or $(\mathcal{P}_{\mu}\gamma^{\mu}+m)\ {}^{-}\phi_{p_{-},n,+1,r}(x)$ and $(\mathcal{P}_{\mu}\gamma^{\mu}+m)\ {}^{-}\phi_{p_{-},n,-1,r}(x)$ are linearly dependent for each sign "+" or "−". It means, in fact, that the spin projections of a particle (+) and an antiparticle (−) can take on only $J_{(d)}$ values. Thus, to construct the complete sets it is sufficient to use only the following solutions:

$${}_{+}^{-}\psi_{p_{-},n,r}(x)=(\mathcal{P}_{\mu}\gamma^{\mu}+m)\ {}_{+}^{-}\phi_{p_{-},n,+1,r}(x)\ . \tag{16}$$

In particular, if $d=2,3$, there is only one spin projection, with only one spinor $v_{+1,r}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$. To get the case where $E=0$ one has to consider a limit in the solution ${}_{+}^{-}\psi_{p_{-},n,r}(x)$, such that $-(qE)^{-1}\ln(\mp\tilde{\pi}_{-})\rightarrow x_{-}/p_{-}$, and $p_{-}>0$ for $\{{}_{+}\psi_{p_{-},n,r}(x)\}$ whereas $p_{-}<0$ for $\{{}^{-}\psi_{p_{-},n,r}(x)\}$.

Using the solutions (13) we may construct the Dirac spinors in a slightly different way,

$$\begin{aligned}
{}_{+}^{-}\psi_{n,\bar{r}}(x)\ &=(\mathcal{P}_{\mu}\gamma^{\mu}+m)_{+}^{-}\phi_{n,\bar{r}}(x)\ , \\
{}_{+}^{-}\phi_{n,\bar{r}}(x)\ &=\left({}_{+}^{-}\phi_{n,+1,r}(x)\ +\mathrm{sgn}\left(qH_{(d-1)/2}\right)_{+}^{-}\phi_{n,-1,r}(x)\ \right),\ \bar{r}=\left(r_{1},r_{2},\dots,r_{(d-3)/2},\mathrm{sgn}\left(qH_{(d-1)/2}\right)\right),
\end{aligned} \tag{17}$$

where

$$R_{j}\ {}_{+}^{-}\phi_{n,\bar{r}}(x)\ =\bar{r}_{j}\ {}_{+}^{-}\phi_{n,\bar{r}}(x),\ j=1,\ 2,\dots,(d-1)/2\ ,$$

and ${}_{+}^{-}\phi_{n,\bar{r}}(x)$ are eigenvectors of the time-independent operator $m^{2}-\mathcal{P}^{i}\mathcal{P}_{i}+\frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}$.

The solutions ${}_{+}\psi_{n,\bar{r}}(x)$ from (17) describe particles (+) and ${}^{-}\psi_{n,\bar{r}}(x)$ describe antiparticles (−) in any time instant. They form a complete and orthonormal set of solutions. There is an interesting asymmetry in energy spectrums of the such particles and antiparticles in odd dimensions. If all quantum numbers $n_j = 0$, and $r_j = \mathrm{sgn}\,(qH_j)$, then $\gamma \mathbf{P}_+^- \psi_{n,\bar{r}}(x) = 0$, and the ground state of the particle has energy ${}_+\varepsilon_0 = m$, but the energy of the ground state of the antiparticle is dependent of the magnetic field and different: $|{}^-\varepsilon_0| = \sqrt{m^2 + 2\min|qH_j|}$.

In the $d = 4$ case the solutions similar to (16) were found first in [7]. In case $E = 0$ they coincide with ones in [16], and if $H = 0$ they coincide with well-known Wolkov form [17]. In the case $E = 0$, the solutions form a complete and orthonormal set, moreover ${}_+\psi_{p_-,n,r}(x)$ describe particles and ${}^-\psi_{p_-,n,r}(x)$ describe antiparticles in any time instant. It is no longer if $E \neq 0$. However, solutions (16) can be used to construct new complete sets which can be classified by the correct way.

One can form two complete and orthonormal sets of the solutions:$\left\{ {}_\pm\psi_{p_-,n,r}(x) \right\}$ and ${}^\pm\psi_{p_-,n,r}(x)$ using (16) and additional sets

$$
\begin{aligned}
{}^+\psi_{p_-,n,r}(x) &= \Theta(\pi_-) \left({}_+\psi(x) G \left({}_+|{}^+ \right) \right)_{p_-,n,r} , \\
{}_-\psi_{p_-,n,r}(x) &= \Theta(-\pi_-) \left({}^-\psi(x) G \left({}_-|{}^- \right)^\dagger \right)_{p_-,n,r} ,
\end{aligned}
\tag{18}
$$

where $G\left({}_\zeta|{}^{\zeta'} \right)$ obey the relations (7) and, in particular, $G\left({}_+|{}^- \right)$ are decomposition coefficients of ${}^-\psi_{p_-,n,r}(x)$ solutions in ${}_+\psi_{p_-,n,r}(x)$ solutions,

$$
G\left({}_+|{}^- \right)_{p_-,n,r,p'_-,n',r'} = ({}_+\psi_{p_-,n,r}, {}^-\psi_{p'_-,n',r'}). \tag{19}
$$

The corresponding Klein-Gordon solutions follow from (11) by the replacement $u_{\xi,r}(x_-) = \exp\left\{ -\frac{1}{2}\ln\left(\mp\tilde{\pi}_- \right) \right\}$.

The orthonormality of all the solutions can be verified by using the following integral transformations:

$$
{}_\pm\psi_{p_-,n,r}(x) = (2\pi qE)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_D, p_-) {}_\pm\psi_{p_D,n,r}(x) dp_D, \tag{20}
$$

$$
{}_\pm\psi_{p_D,n,r}(x) = (2\pi qE)^{-1/2} \int_{-\infty}^{+\infty} M(p_D, p_-) {}_\pm\psi_{p_-,n,r}(x) dp_-, \tag{21}
$$

$$
M(p_D, p_-) = \exp\left\{ -\frac{i}{4qE}((p_- - 2p_D)^2 - 2(p_D)^2) \right\},
$$

$$
\int_{-\infty}^{+\infty} M^*(p_D, p'_-) M(p_D, p_-) dp_D = 2\pi qE\delta(p_- - p'_-).
$$

The same relation is valid for functions with $(\pm)$ indices above.

The saddle points $\pi_- = -2(qEx_0 - p_D)$ give the main contribution to the integrals (21) with ${}^{-}_{+}\psi_{p_-,n,r}(x)$ functions at $x_0 \to \pm\infty$ (this was first found in Ref. [18] for the $d = 4$ case without magnetic field). Since the plane wave vanishes in ${}^{-}_{+}\psi_{p_-,n,r}(x)$ at $\pi_- \to \pm\infty$, relations (21) reduce at $x_0 \to \pm\infty$ to the space-time uniform field case and can be presented by use of formulas from [19] as follows

$$
\begin{aligned}
{}^{-}_{+}\psi_{p_D,n,r}(x) &= (\mathcal{P}_\mu\gamma^\mu + m)\ {}^{-}_{+}\phi_{p_D,n,r}(x)\ ,\ x_0 \to \pm\infty, \\
{}^{-}_{+}\phi_{p_D,n,r}(x) &= \ {}^{-}_{+}\phi_{n,r}(x^0)\exp\left(-ip_Dx^D\right)\ \phi_{n,r}(x_\perp)v_{+1,r}, \\
{}^{-}_{+}\phi_{n,r}(x^0) &= CD_{\nu-1}[\pm(1-i)\xi],
\end{aligned}
\tag{22}
$$

$$
\xi = \left(qEx^0 - p_D\right)/\sqrt{qE},\ \nu = i\lambda/2,\ C = (4\pi qE)^{-1/2}\exp\{(-\pi/2 + i\ln 2)\lambda/4\}(-i),
$$

where the function $\phi_{n,r}(x_\perp)$ was defined in (11), and, $D_\nu(z)$ is the Weber parabolic cylinder function [20]. Such solutions were discussed in [13]. One can now verify the orthonormality relations of the sets$\{{}_{+}\psi_{p_D,n,r}(x)\}$ and $\{{}^{-}\psi_{p_D,n,r}(x)\}$. Using transformation (20) one gets the orthonormality relations of sets $\left\{{}_{+}\psi_{p_-,n,r}(x)\right\}$ and $\left\{{}^{-}\psi_{p_-,n,r}(x)\right\}$ as well. Using the explicit forms of solutions (16), (18), and relations (7) one can derive the following relations,

$$
\begin{aligned}
{}^{+}\psi(x)G\left({}_{+}|{}^{+}\right)^{\dagger} &= {}_{+}\psi(x) - {}^{-}\psi(x)G\left({}_{+}|{}^{-}\right)^{\dagger}, \\
{}_{-}\psi(x)G\left({}_{-}|{}^{-}\right) &= {}^{-}\psi(x) -_{+}\psi(x)G\left({}_{+}|{}^{-}\right).
\end{aligned}
\tag{23}
$$

By means of the latter one can get the orthonormality relations for both sets of the solutions,

$$
\begin{aligned}
\left({}_{\zeta}\psi_{p_-,n,r},\ {}_{\zeta'}\psi_{p'_-,n',r'}\right) &= \delta_{\zeta\zeta'}\delta_{rr'}\delta_{nn'}\delta(p_- - p'_-), \\
\left({}^{\zeta}\psi_{p_-,n,r},\ {}^{\zeta'}\psi_{p'_-,n',r'}\right) &= \delta_{\zeta\zeta'}\delta_{rr'}\delta_{nn'}\delta(p_- - p'_-),
\end{aligned}
\tag{24}
$$

where $\zeta, \zeta' = \pm$, $\delta_{nn'}$ is the Kronecker symbol for the discrete spectrum and the δ-function for the continuous one. Here $r = r' = +1$ if $d = 2, 3$.

To solve a problem of the $(\pm)$ classification of the solutions one needs to study their asymptotic behavior at $x_0 \to \pm\infty$. From the asymptotic forms (22) it follows [13] that the asymptotic of the quasienergies of these solution: ${}_+\varepsilon = -qEx^0$, is positive and ${}^-\varepsilon = -qEx^0$ is negative. Thus, the solutions ${}_+\psi_{p_D,n,r}(x)$ describe particles at $x_0 \to -\infty$, and the solutions ${}^-\psi_{p_D,n,r}(x)$ describe antiparticles at $x_0 \to +\infty$. Since the solutions ${}_-\psi_{p_-,n,r}(x)$ are orthogonal to ${}_+\psi_{p_-,n,r}(x)$, they describe antiparticles at $x_0 \to -\infty$, and solutions ${}^+\psi_{p_-,n,r}(x)$ describe particles at $x_0 \to +\infty$ since they are orthogonal to ${}^-\psi_{p_-,n,r}(x)$. One can verify this by taking into account that the main contribution to integrals (21) at $x_0 \to \pm\infty$ for ${}^+_-\psi_{p,r}(x)$ is given by point $\pi_- = 0$. In this limit the contribution of the plane wave does not depend on x and the results of transformation (21) are proportional to a superposition of the solutions in a constant uniform field [20],

$$
\begin{aligned}
{}^+_-\psi_{p_D,n,r}(x) &= \sum_{n,r} a_{n'r'} \left(\mathcal{P}_\mu \gamma^\mu + m\right)\ {}^+_-\phi_{p_D,n',r'}(x)\ ,\ \ x_0 \to \pm\infty, \\
{}^+_-\phi_{p_D,n,r}(x) &= \ {}^+_-\phi_{n,r}(x^0) \exp\left(-ip_D x^D\right)\ \phi_{n,r}(x_\perp) v_{+1r}, \\
{}^+_-\phi_{n,r}(x^0) &= C D_{-\nu}[\pm(1+i)\xi],
\end{aligned} \tag{25}
$$

where a_{nr} are some coefficients dependent on the plane-wave form. From the asymptotic representations (25) one can see [13] that the asymptotic of the quasienergies of these solutions: ${}^+\varepsilon = qEx^0$ is positive and ${}_-\varepsilon = qEx^0$ is negative.

Both sets $\{{}_\pm\psi_{p_-,n,r}(x)\}$ and $\{{}^\pm\psi_{p_-,n,r}(x)\}$ are orthonormal and complete at any time instant. The form of the commutation function, which will be present in the next section, can serve as direct proof of the last statement. In $d = 4$ such complete sets of the solutions were first found in [8]. If the plane wave is absent and $E = 0$ one can get the solutions with defined energies from ${}^-_+\psi_{p_D,n,r}(x)$ (22) considering the next limit in ${}^-_+\phi_{n,r}(x^0)$:

$$
\begin{aligned}
{}^-_+\phi_{n,r}(x^0) &\to \left[{}_\mp\varepsilon_{nr}({}_\mp\varepsilon_{nr} + p_D)\right]^{-1} \exp\left(\pm i \left|{}_\mp\varepsilon_{nr}\right| x^0\right), \\
\left|{}_\mp\varepsilon_{nr}\right| &= \sqrt{m^2 + \sum_{j=1}^{[(d-1)/2]} \omega_j + \omega_0'},\ \ \omega_0' = \begin{cases} p_D^2,\ d \text{ is even} \\ 0,\ d \text{ is odd} \end{cases},
\end{aligned}
$$

where ω_j is defined in (12). In this case one can choose ${}^+\psi_{p_D,n,r}(x) ={}_+ \psi_{p_D,n,r}(x)$ and ${}_-\psi_{p_D,n,r}(x) ={}^- \psi_{p_D,n,r}(x)$.

Let us select some important properties of the solution. One can see that the matrix elements $G\left({}_{\zeta}|{}^{\zeta'}\right)$ are diagonal with respect to continuous quantum numbers and spin quantum numbers,

$$G\left({}_{\zeta}|{}^{\zeta'}\right)_{p_-,n,r,p'_-,n',r'} = \delta_{rr'}\delta(p_- - p'_-)\delta(p_1 - p'_1)\dots\delta(p_{2[(d-1)/2]-1} - p'_{2[(d-1)/2]-1})\; g\left({}_{\zeta}|{}^{\zeta'}\right)_{nn'}\,. \tag{26}$$

The solutions ${}_{+}^{-}\psi_{p_-,n,r}(x)$ satisfy the orthonormality conditions on the null-plane,

$$\int {}_{+}^{-}\bar{\psi}_{p_-,n,r}(x)\Xi_-\; {}_{+}^{-}\psi_{p'_-,n',r'}(x)dx_+dx^1...dx^{D-1} = \delta_{rr'}\delta_{nn'}\delta(p_- - p'_-),\; \mp\pi_- > 0. \tag{27}$$

Since ${}^{+}\psi_{p_-,n,r}(x) = 0$ for $\pi_- < 0$ and relations (23) take place, one gets the following representations for $G\left({}_{+}|{}^{-}\right)G\left({}_{+}|{}^{-}\right)^{\dagger}$ and $G\left({}_{+}|{}^{-}\right)^{\dagger}G\left({}_{+}|{}^{-}\right)$ matrices:

$$\begin{aligned}
\left(G\left({}_{+}|{}^{-}\right)G\left({}_{+}|{}^{-}\right)^{\dagger}\right)_{p_-,n,r,p'_-,n',r'} &= \int {}_{+}\bar{\psi}_{p_-,n,r}(x)\Xi_-\; {}_{+}\psi_{p'_-,n',r'}(x)dx_+dx^1...dx^{D-1},\; \pi_- < 0,\\
\left(G\left({}_{+}|{}^{-}\right)^{\dagger}G\left({}_{+}|{}^{-}\right)\right)_{p_-,n,r,p'_-,n',r'} &= \int {}^{-}\bar{\psi}_{p_-,n,r}(x)\Xi_-\; {}^{-}\psi_{p'_-,n',r'}(x)dx_+dx^1...dx^{D-1},\; \pi_- > 0.
\end{aligned} \tag{28}$$

Thus one can calculate ${}_{+}\mathcal{D} = g\left({}_{+}|{}^{-}\right)g\left({}_{+}|{}^{-}\right)^{\dagger}$ and ${}^{-}\mathcal{D} = g\left({}_{+}|{}^{-}\right)^{\dagger}g\left({}_{+}|{}^{-}\right)$ matrices using the following integrals with the solutions of the squared Dirac equation:

$${}_{+}^{-}\mathcal{D}_{nn'} = \int\; {}_{+}^{-}\phi^{\dagger}_{p_-,n,+1,r}(x)\; {}_{+}^{-}\phi_{p_-,n',+1,r}(x)dx^2dx^4...dx^{2[(d-1)/2]},\; \mp\pi_- < 0. \tag{29}$$

III. GREEN FUNCTIONS

The perturbation theory with respect to the radiative interaction for the matrix elements of the processes also has the usual Feynman structure in an external field creating pairs [5,21,3]. The Feynman diagrams have to be calculated by means of the causal propagator

$$S^c(x,x') = c_v^{-1}i < 0,out|T\psi(x)\bar{\psi}(x')|0,in >,\;\; c_v =< 0,out|0,in >\,, \tag{30}$$

where $\psi(x)$ is quantum spinor field in the generalized Furry representation, satisfying the Dirac equation (1), $|0,in>$ and $|0,out>$ are the initial and the final vacuum in this representation, and c_v is the vacuum to vacuum transition amplitude. The propagator $S^c(x,x')$ obeys the equation

$$(\mathcal{P}_\mu \gamma^\mu - m) \; S^c(x,x') = -\delta^{(d)}(x-x') \; , \tag{31}$$

and is a Green function of the equation. Another important singular function is the commutation function

$$S(x,x') = i \left[\psi(x), \bar{\psi}(x') \right]_+ . \tag{32}$$

It obeys the homogeneous Dirac equation (1) and the initial condition

$$S(x,x')|_{x0=x'_0} = i\gamma^0 \delta(\mathbf{x} - \mathbf{x}'). \tag{33}$$

The commutation function $S(x,x')$ is at the same time the propagation function of the Dirac equation, i.e. it connects solutions of the equation in two different time instants.

QED with unstable vacuum has a number of peculiarities. Thus, for instance, in the calculation of the expectation values and the total probabilities Green functions of different types from (30) appear [5,21,3]:

$$\begin{aligned}
S^c_{in}(x,x') &= i < 0, in|T\psi(x)\bar{\psi}(x')|0, in >, \\
S^-_{in}(x,x') &= i < 0, in|\psi(x)\bar{\psi}(x')|0, in >, \\
S^+_{in}(x,x') &= i < 0, in|\bar{\psi}(x')\psi(x)|0, in >, \\
S^{\bar{c}}_{in}(x,x') &= i < 0, in|\psi(x)\bar{\psi}(x')T|0, in >, \\
S^c_{out}(x,x') &= i < 0, out|T\psi(x)\bar{\psi}(x')|0, out >,
\end{aligned} \tag{34}$$

where the symbol of the T-product acts on both sides: it orders the field operators to the right of its and antiorders them to the left. The functions $S^c_{in}(x,x')$, $S^c_{out}(x,x')$ obey Eq. (31), $S^{\mp}(x,x')$ satisfies Eq. (1) and $S^{\bar{c}}_{in}(x,x')$ obeys the equation

$$(\mathcal{P}_\mu \gamma^\mu - m) \; S^{\bar{c}}_{in}(x,x') = \delta^{(d)}(x-x') \; . \tag{35}$$

As well, all these different kinds of the Green functions are used to represent various matrix elements of operators of the current and energy-momentum tensor, and effective action beginning with zeroth order with respect to radiative interaction.

Solutions (16) and (18) with quantum number p_- are especially adapted to calculate all the Green functions. One can express the Green functions via solutions (16) and (18) [5,21,3]:

$$S^c(x,x') = \theta(x_0 - x'_0)\, S^-(x,x') - \theta(x'_0 - x_0)\, S^+(x,x')\,, \tag{36}$$

$$S(x,x') = S^-(x,x') + S^+(x,x')\,, \tag{37}$$

$$\begin{aligned} S^-(x,x') &= i\int_{-\infty}^{+\infty} dp_- \sum_{nr\{n'_j\}} {}^+\psi_{p_-,n,r}(x)\, g\left({}_+|{}^+\right)^{-1}_{nn'}\ {}_+\bar{\psi}_{p_-,n',r}(x')\,, \\ S^+(x,x') &= i\int_{-\infty}^{+\infty} dp_- \sum_{nr\{n'_j\}} {}_-\psi_{p_-,n,r}(x)\left[g\left({}_-|{}^-\right)^{-1}\right]^*_{nn'}\ {}^-\bar{\psi}_{p_-,n',r}(x')\,, \end{aligned} \tag{38}$$

$$S^c_{in}(x,x') = \theta(x_0 - x'_0)\, S^-_{in}(x,x') - \theta(x'_0 - x_0)\, S^+_{in}(x,x')\,, \tag{39}$$

$$S^{\bar{c}}_{in}(x,x') = \theta(x'_0 - x_0)\, S^-_{in}(x,x') - \theta(x_0 - x'_0)\, S^+_{in}(x,x')\,, \tag{40}$$

$$S^{\mp}_{in}(x,x') = i\int_{-\infty}^{+\infty} dp_- \sum_{nr}\ {}_\pm\psi_{p_-,n,r}(x)_\pm\, \bar{\psi}_{p_-,n,r}(x')\,, \tag{41}$$

$$S^c_{out}(x,x') = \theta(x_0 - x'_0)\, S^-_{out}(x,x') - \theta(x'_0 - x_0)\, S^+_{out}(x,x')\,, \tag{42}$$

$$S^{\mp}_{out}(x,x') = i\int_{-\infty}^{+\infty} dp_- \sum_{nr}\ {}^\pm\psi_{p_-,n,r}(x)^\pm\, \bar{\psi}_{p_-,n,r}(x')\,. \tag{43}$$

where all $p'_j = p_j$, the symbol $\sum_{nr}$ means the summation over all discrete quantum numbers n_j, r_j and the integration over all continuous p_j, and the symbol $\sum_{\{n_j\}}$ means the summation over all discrete quantum numbers n_j only. Using the relations between the Green functions and between the matrices $G\left({}_\zeta|{}^{\zeta'}\right)$ one can present the functions $S^\mp$, $S^\mp_{in}$ and $S^\mp_{out}$ as follows

$$\pm S^\mp(x,x') = S^c(x,x') \pm \theta(\mp(x_0 - x'_0))S(x,x') \quad , \tag{44}$$

$$\pm S^\mp_{in}(x,x') = S^c_{in}(x,x') \pm \theta(\mp(x_0 - x'_0))S(x,x') \quad , \tag{45}$$

$$\pm S^\mp_{out}(x,x') = S^c_{out}(x,x') \pm \theta(\mp(x_0 - x'_0))S(x,x') \quad , \tag{46}$$

$$S^c_{in}(x,x') = S^c(x,x') - S^a(x,x') \quad , \tag{47}$$

$$S^c_{out}(x,x') = S^c(x,x') - S^p(x,x') \quad , \tag{48}$$

$$S^a(x,x') = -i\int_{-\infty}^{+\infty} dp_- \sum_{nr\{n'_j\}} {}_-\psi_{p_-,n,r}(x) \left[g({}_+|^-)g({}_-|^-)^{-1}\right]^\dagger_{nn'} {}_+\bar{\psi}_{p_-,n',r}(x') \quad , \tag{49}$$

$$S^p(x,x') = i\int_{-\infty}^{+\infty} dp_- \sum_{nr\{n'_j\}} {}^+\psi_{p_-,n,r}(x) \left[g({}_+|^+)^{-1}g({}_+|^-)\right]_{nn'} {}^-\bar{\psi}_{p_-,n',r}(x') \quad . \tag{50}$$

To calculate all the types of Green functions it is sufficient to take sums in $S^\pm(x,x')$ and $S^{a,p}(x,x')$ only. That will be done below.

It follows from (18)

$$S^\pm(x,x') = \int_{-\infty}^{+\infty} \Theta\left(\mp\pi_-\right) {}^-_+Y(x,x',p_-)dp_-, \tag{51}$$

$${}^-_+Y(x,x',p_-) = i\sum_{nr} {}^-_+\psi_{p_-,n,r}\left(x\right)\ {}^-_+\overline{\psi}_{p_-,n,r}\left(x'\right). \tag{52}$$

Also, taking into account Eqs. (23) one gets

$$\begin{aligned} S^a(x,x') &= -\int_{-\infty}^{+\infty} \Theta\left(-\pi_-\right)_+ Y(x,x',p_-)dp_-, \\ S^p(x,x') &= \int_{-\infty}^{+\infty} \Theta\left(\pi_-\right)^- Y(x,x',p_-)dp_-, \end{aligned} \tag{53}$$

Due to the fact that ${}_+\psi_{p_-,n,r}\left(x\right)$ and ${}^-\psi_{p_-,n,r}\left(x\right)$ have similar form (16), the sums in (52) can be taken in the same way. Since

$$\sum_r v_{+1,r}v^+_{+1,r} = \Xi_+ \text{ if } d>3, \text{ and } v_{+1}v^+_{+1} = \Xi_+ \text{ if } d=2,3,$$

the summation over the spin quantum numbers can be done in (52) to get

$$\begin{aligned} {}^-_+Y(x,x',p_-) &= \left[\gamma^0 + (\gamma_\perp\mathcal{P}_\perp + m)\frac{1}{\pi_-}\right]\Xi_+ \exp\left(-i\frac{q}{2}\sigma^{\mu\nu}F^\perp_{\mu\nu}\ {}^-_+a\right)\left[\gamma^0 + \left(-\gamma_\perp\mathcal{P}'^*_\perp + m\right)\frac{1}{\pi'_-}\right] \\ &\gamma^0 \exp\left\{\frac{1}{2}\left[\ln\left(\mp\tilde{\pi}_-\right) + \left(\ln\left(\mp\tilde{\pi}'_-\right)\right)^*\right]\right\}\cdot\sqrt{qE}\,{}^-_+\tilde{f}^{(0)}\left(x,x',p_-\right), \end{aligned} \tag{54}$$

$${}^-_+\tilde{f}^{(0)}\left(x,x',p_-\right) = i\sum_n {}^-_+\varphi_{p_-,n,r}\left(x\right)\ {}^-_+\varphi^*_{p_-,n,r}\left(x'\right), \tag{55}$$

$$\begin{aligned} {}^-_+\varphi_{p_-,n,r}\left(x\right) &= \exp\left\{-\frac{1}{2}\ln\left(\mp\tilde{\pi}_-\right)\right\}\ {}^-_+\phi_{p_-,n,r}(x^0,x^D)\phi_{n,r}(x_\perp), \\ {}^-_+a &= (2qE)^{-1}\left[\left(\ln\left(\mp\tilde{\pi}'_-\right)\right)^* - \ln\left(\mp\tilde{\pi}_-\right)\right], \\ \tilde{\pi}'_- &= \tilde{\pi}_- + \sqrt{qE}y_-,\ y_\mu = x_\mu - x'_\mu, \\ \mathcal{P}_{\perp\mu} &= 0 \text{ if } \mu=0,D, \quad \mathcal{P}_{\perp\mu} = \mathcal{P}_\mu \ \text{ if } k=1,\dots,D-1\ , \\ \mathcal{P}'^*_\mu &= -i\frac{\partial}{\partial x'^\mu} - q\,A_\mu(x'). \end{aligned} \tag{56}$$

It is convenient to make the replacement

$$u=(2qE)^{-1}\left[\left(\ln\left(\mp\tilde{\pi}'_-\right)\right)^*-\ln\left(\mp\tau\right)\right]$$

in the functions ${}^-_+K(x_-)$ and ${}^-_+J(x_-)$, and the one

$$u=(2qE)^{-1}\left[\left(\ln\left(\tilde{\pi}'_-\right)\right)^*-\ln\tau\right]$$

in the functions ${}^-_+K^*(x'_-)$ and ${}^-_+J^*(x'_-)$ (from ${}^-_+\phi_{p_-,n,r}(x^0,x^D)$ and ${}^-_+\phi^*_{p_-,n,r}(x'^0,x'^D)$ defined in (11)). We do not discuss the integration paths over the variable u, since it is natural to consider the plane-wave potential $f(x_-)$ to be an entire function on x_- , in that case the integrals ${}^-_+K(x_-)$, ${}^-_+J(x_-)$, ${}^-_+K^*(x'_-)$ and ${}^-_+J^*(x'_-)$ are well defined by the integration limits. For different forms of the plane-wave potential the integration paths over u are no longer arbitrary, however, they easily are extracted from the integration paths shown in Fig.1 and Fig.2. Then, one can present some combinations involved in (55) in the following way:

$$\begin{aligned}
{}^-_+K\left(x_-\right)-{}^-_+K^*\left(x'_-\right)&=l\left({}^-_+a\right)+2\left(e^{2qF{}^-_+a}-1\right)\int_{{}^-_+b}^0 e^{-2qFu}qf\left(x_-\left(u\right)\right)du,\\
{}^-_+K\left(x_-\right)+{}^-_+K^*\left(x'_-\right)&=l\left({}^-_+a\right)+2\left(e^{2qF{}^-_+a}+1\right)\int_{{}^-_+b}^0 e^{-2qFu}qf\left(x_-\left(u\right)\right)du,\\
{}^-_+J\left(x_-\right)-{}^-_+J^*\left(x'_-\right)&=\Phi\left({}^-_+a\right)+2\int_0^{{}^-_+a}qf\left(x_-\left(u\right)\right)e^{2qFu}du\;qF\int_{{}^-_+b}^0 e^{-2qFu'}qf\left(x_-\left(u'\right)\right)du',
\end{aligned}\tag{57}$$

where

$$\begin{aligned}
\Phi\left({}^-_+a\right)&=\int_0^{{}^-_+a}qf\left(x_-\left(u\right)\right)\left[qf\left(x_-\left(u\right)\right)+qFl\left(u\right)\right]du,\\
l\left(u\right)&=2\int_0^u e^{2qF(u-u')}qf\left(x_-\left(u'\right)\right)du',\\
{}^-_+b&=-\infty-i\pi/\left(2qE\right)\Theta\left(\pm\pi'_-\right),\;\;x_-(u)=x'_-+y_-\frac{1-\exp\left\{-2qEu\right\}}{1-\exp\left\{-2qE{}^-_+a\right\}}.
\end{aligned}\tag{58}$$

The summation over n_j in (55) can be performed using the Möller formula [20]. After integration over p_j in (55), applying the operators $\exp\left\{-i{}^-_+K\left(x_-\right)\pi_\perp\right\}$ and $\exp\left\{-i{}^-_+K\left(x_-\right)\pi'_\perp\right\}^*$, and using relations (57), we obtain

$$\begin{aligned}
{}^-_+\tilde{f}^{(0)}\left(x,x',p_-\right)&=\exp\left\{iq\Lambda-im^2\,{}^-_+a-\right.\\
&\left.-\frac{1}{2}\left[\ln\left(\mp\tilde{\pi}_-\right)+\left(\ln\left(\mp\tilde{\pi}'_-\right)\right)^*\right]-i\frac{\pi_-+\pi'_-}{4}y_+\right\}h({}^-_+a\;),
\end{aligned}\tag{59}$$

$$h({}^-_+a\;)=\exp\left\{i\Phi({}^-_+a)-i\frac{1}{4}(y+l({}^-_+a))qF^\perp\coth(qF^\perp\,{}^-_+a)(y+l({}^-_+a))+\frac{i}{2}yqF^\perp l(s)\right\}Z_{(d)},\tag{60}$$

where

$$
\begin{aligned}
Z_{(d)} &= c_d \prod_{j=1}^{(d-2)/2} \left(\frac{qH_j}{\sin(qH_j s)} \right), \qquad d \text{ is even}, \\
Z_{(d)} &= c_d s^{-1/2} \prod_{j=1}^{(d-3)/2} \left(\frac{qH_j}{\sin(qH_j s)} \right), \quad d \text{ is odd}, \\
c_d &= (4\pi)^{-d/2} \exp\left\{-i\pi(d-4)/4\right\} \,,
\end{aligned} \tag{61}
$$

and

$$
\Lambda = -\int_{x'}^{x} \left(A_\mu^E + A_\mu^H \right) dx^\mu . \tag{62}
$$

Here $A_\mu^E + A_\mu^H$ is a potential of the constant uniform field $F_{\mu\nu}$, and the integral is taken along the line.

Let us remark that after convenient gauge transformation of electric constant field potentials the functions ${}^{-}_{+}\tilde{f}^{(0)}(x, x', p_-)$ obeys the Klein-Gordon equation,

$$
\left(\pi_- 2i \frac{\partial}{\partial x_-} - iqE + \mathcal{P}_\perp^2 - m^2 \right) \exp\left\{ -\frac{iqE}{2} \left(\frac{x_-^2}{2} - x_D^2 \right) \right\} {}^{-}_{+}\tilde{f}^{(0)}(x, x', p_-) = 0. \tag{63}
$$

Taking into account the relations

$$
\begin{aligned}
\pi_\perp e^{iq\Lambda} &= e^{iq\Lambda} \left(i \frac{\partial}{\partial x_\perp} + \frac{1}{2} qFy_\perp \right), \\
\pi_\perp'^{*} e^{iq\Lambda} &= e^{iq\Lambda} \left(-i \frac{\partial}{\partial x'_\perp} - \frac{1}{2} qFy_\perp \right),
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
\exp\left(-iq\sigma^{0D} F_{0D} \, {}^{-}_{+}a \right) &= \cosh\left(qE \, {}^{-}_{+}a \right) - i\sigma^{0D} \sinh\left(qE \, {}^{-}_{+}a \right), \\
\exp\left(-i\frac{q}{2} \sigma^{\mu\nu} F_{\mu\nu}^{(j)} \, {}^{-}_{+}a \right) &= \cos\left(qH_j \, {}^{-}_{+}a \right) + iR_j \sin\left(qH_j \, {}^{-}_{+}a \right),
\end{aligned} \tag{65}
$$

where

$$
F_{\mu\nu}^{(j)} = H_j (\delta_\mu^{2j} \delta_\nu^{2j-1} - \delta_\nu^{2j} \delta_\mu^{2j-1}),
$$

and matrix R_j is defined by (15), one can get relation

$$
\begin{aligned}
&\exp\left(-i\frac{q}{2} \sigma^{\mu\nu} F_{\mu\nu}^{\perp} \, {}^{-}_{+}a \right) \gamma\pi_\perp'^{*} \, {}^{-}_{+}\tilde{f}^{(0)}(x, x', p_-) = \\
&\left(\gamma\pi_\perp + \gamma qF^\perp l \left({}^{-}_{+}a \right) \right) \exp\left(-i\frac{q}{2} \sigma^{\mu\nu} F_{\mu\nu}^{\perp} \, {}^{-}_{+}a \right) \, {}^{-}_{+}\tilde{f}^{(0)}(x, x', p_-) \,.
\end{aligned} \tag{66}
$$

By using formulas (65), (66) and Eq. (63) one can transform expression (54) to the following form,

$$
\begin{aligned}
{}_{+}^{-}Y(x,x',p_-) &= \mp(\gamma\mathcal{P}+m)\ {}_{+}^{-}\tilde{f}(x,x',p_-)\,, &(67)\\
{}_{+}^{-}\tilde{f}(x,x',p_-) &= \Big[\exp\left(-i\frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}\ {}_{+}^{-}a\right) \mp \frac{1}{2}(\gamma^0-\gamma^D)\gamma\int_0^{{}_{+}^{-}a} e^{qF({}_{+}^{-}a-2u)}q\frac{df(x_-(u))}{du}du \\
&\quad \exp\left\{-\frac{1}{2}\left[\ln(\mp\tilde{\pi}_-)+\left(\ln\left(\mp\tilde{\pi}'_-\right)\right)^*\right]\right\}\Big]\ {}_{+}^{-}\tilde{f}^{(0)}(x,x',p_-)\,. &(68)
\end{aligned}
$$

In the external field under consideration the real proper time S is a function of π_- and π'_- because the classical equation of motion has a form $\pi_-^{-1}dx_- = m^{-1}dS$. Thus if $y_- \neq 0$ one can transform the p_- integration in Green functions into integration over the Fock-Schwinger proper time by making the change of the variable,

$$
s = {}_{+}^{-}a\ . \tag{69}
$$

Then, one gets the following representations for the Green functions:

$$
\begin{aligned}
S^{\mp,a,p}(x,x') &= (\gamma\mathcal{P}+m)\Delta^{\mp,a,p}(x,x')\ ,\\
\mp\Delta^{\pm}(x,x') &= \int_{\Gamma_c} f(x,x',s)ds - \Theta(\pm y_-)\int_{\Gamma_c-\Gamma_2-\Gamma_1} f(x,x',s)ds\ , &(70)\\
\Delta^{a}(x,x') &= \int_{\Gamma_a} f(x,x',s)ds + \Theta(y_-)\int_{\Gamma_2+\Gamma_3-\Gamma_a} f(x,x',s)ds\ . &(71)\\
\Delta^{p}(x,x') &= \int_{\Gamma_a} f(x,x',s)ds + \Theta(-y_-)\int_{\Gamma_1^a} f(x,x',s)ds\ . &(72)
\end{aligned}
$$

All the contours of the integrals are shown on Fig. 3. The contours Γ_c and Γ_1 are placed below the singular points on the real axis everywhere outside of the origin, and

$$
\begin{aligned}
f(x,x',s) &= \Big[\exp\left(-i\frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}s\right)\\
&+(n\gamma)\gamma\int_0^s e^{qF(s-2u)}\frac{df((nx(u)))}{du}du\frac{\sinh(qEs)}{E(ny)}\Big]\ f^{(0)}(x,x',s), &(73)\\
f^{(0)}(x,x',s) &= \exp\{iq\Lambda\}\frac{qE}{\sinh(qEs)}\exp\{-im^2s+i\Phi(s)\\
&-i\frac{1}{4}(y+l(s))qF\coth(qFs)(y+l(s))+\frac{i}{2}yqFl(s)\}Z_{(d)}, &(74)
\end{aligned}
$$

where $Z_{(d)}$ is defined in (61). The function $f^{(0)}(x,x',s)$ has two singular points on the

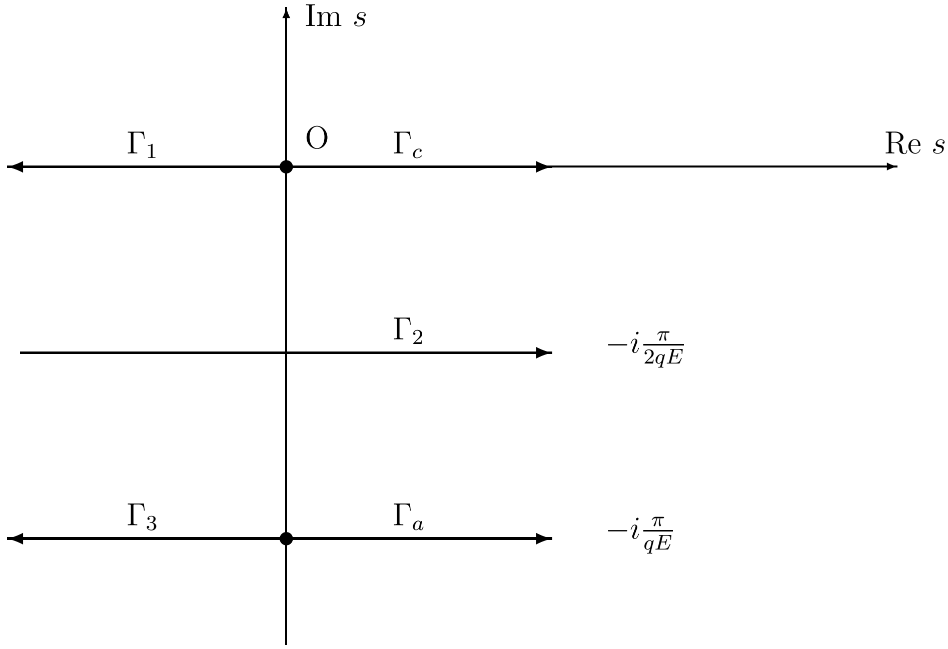

FIG. 3. Contours of integration $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_c, \Gamma_a$.

complex region between the contours $\Gamma_c - \Gamma_1$ and $\Gamma_a - \Gamma_3$. They are situated at the imaginary axis: $s_0 = 0$, and $qEs_1 = -i\pi$. One can transform the contours $\Gamma_c - \Gamma_2 - \Gamma_1$ into Γ and $\Gamma_2 + \Gamma_3 - \Gamma_a$ into Γ_1^a (see Fig. 4) with radii tending to zero. Since

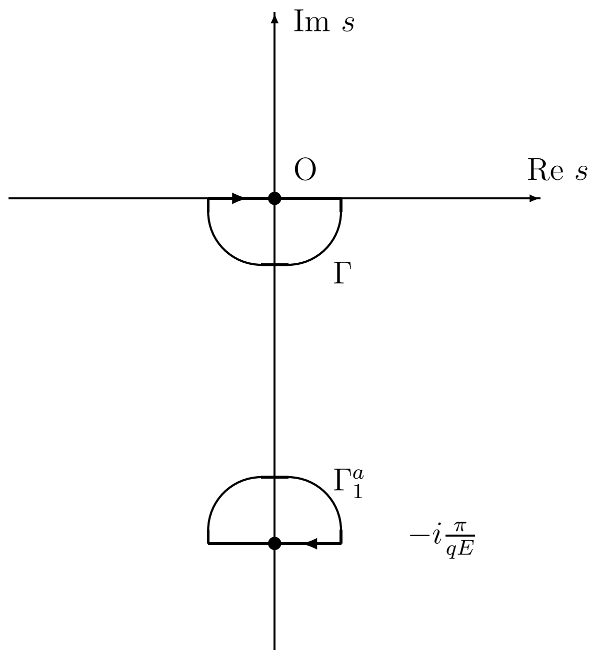

FIG. 4. Contours of integration Γ, Γ_1^a.

$$\int_{\Gamma} f(x,x',s)ds = 0 \text{ if } y_\mu y^\mu < 0\ , \quad \int_{\Gamma_1^a} f(x,x',s)ds = 0 \text{ if } y_0^2 > y_D^2\ ,$$

one can rewrite (70),(71) and (72) as follows

$$S^{\mp}(x,x') = (\gamma\mathcal{P}+m)\Delta^{\mp}(x,x'), \tag{75}$$

$$\mp\Delta^{\pm}(x,x') = \int_{\Gamma^c} f(x,x',s)ds - \Theta(\pm y_0)\int_{\Gamma} f(x,x',s)ds\ ,$$

$$S^{a,p}(x,x') = (\gamma\mathcal{P}+m)\Delta^{a,p}(x,x'), \tag{76}$$

$$\Delta^{a}(x,x') = \int_{\Gamma^a} f(x,x',s)ds + \Theta(-y^D)\int_{\Gamma_1^a} f(x,x',s)ds\ ,$$

$$\Delta^{p}(x,x') = \int_{\Gamma^a} f(x,x',s)ds + \Theta(y^D)\int_{\Gamma_1^a} f(x,x',s)ds\ .$$

One can check that these expressions are valid for arbitrary x and x'. To prove this one need to verify that the functions $S^{\pm}(x,x')$ and $S^{a,p}(x,x')$, which are presented via integrals (75) and (76), and by means of representations (51) and (53), are the same solutions of the Dirac equation for any x and x'. Thus it is enough, to check first that expressions (75) and (76) obey the Dirac equation for any x and x'. Then, one has to prove that the Cauchy conditions for distributions (75) and (76) coincide at $x_0 = x_0'$ with (51) and (53) respectively. As to $S^{\pm}(x,x')$, one can use the proper time representations for $S^c(x,x')$ and $S(x,x')$ functions, which follow from (75), (36) and (37),

$$S^{c}(x,x') = (\gamma\mathcal{P}+m)\Delta^{c}(x,x'),$$

$$\Delta^{c}(x,x') = \int_{\Gamma_c} f(x,x',s)ds, \tag{77}$$

$$S(x,x') = (\gamma\mathcal{P}+m)\Delta(x,x'),$$

$$\Delta(x,x') = \text{sign}(x_0 - x_0')\int_{\Gamma} f(x,x',s)ds. \tag{78}$$

One can see that $f(x,x',s)$ obeys the following equations,

$$-i\frac{d}{ds} f(x,x',s) = \left(\mathcal{P}^2 - m^2 - \frac{q}{2}\sigma^{\mu\nu}\mathcal{F}_{\mu\nu}\right) f(x,x',s) \quad , \tag{79}$$

$$\lim_{s\to+0} f(x,x',s) = i\delta^{(d)}(x-x') \quad . \tag{80}$$

Thus $f(x,x',s)$ is the Fock-Schwinger function [22,23]. So, representation (77) of causal Green function has the well-known Schwinger form [22]. The concrete representation (78) of the commutation function has the same form as the general representation [24]. Similar to [24] one can select all singularities of this function and see that it is continuous at $x_0 - x_0'$. In $d = 4$ case one can transform this representation to the Fock form [23]. The analysis of space-time singularities in the integrals (76) over the contour Γ_1^a can be done in a similar manner to [24]. In the $d = 4$ the representations (75) - (78) coincide with the ones found in [8]. To find the proper time representation we used the special behavior of solutions (16). But one has the different form of solutions (17) if d is odd, $E = 0$ and all the imaginary eigenvalues H_j of the field tensor are not equal to zero. In this case since solutions (17) can represent a special case of the $d+1$ general form (16), one does not need to calculate Green functions of the case independently and representations (75), (77) and (78) are valid if one puts $E = 0$, $f((nx)) = 0$ and replaces $qE/\sinh(qEs) \rightarrow qH_{(d-1)/2}/\sin\left(qH_{(d-1)/2}\right)$ in formulas (73) and (74) (the functions $S^{a,p}(x,x')$ are equal to zero).

IV. SOME PHYSICAL APPLICATIONS

All the information about the processes of particle creation, annihilation, and scattering in an external field (without radiative corrections) can be extracted from the matrices $G\left({}_{\zeta}|{}^{\zeta'}\right)$ (6). These matrices define a canonical transformation between in and out creation and annihilation operators in the generalized Furry representation [5,3],

$$
\begin{aligned}
a^{\dagger}(out) &= a^{\dagger}(in)G\left({}_{+}|{}^{+}\right) + b(in)G\left({}_{-}|{}^{+}\right), \\
b(out) &= a^{\dagger}(in)G\left({}_{+}|{}^{-}\right) + b(in)G\left({}_{-}|{}^{-}\right).
\end{aligned} \tag{81}
$$

Here $a^{\dagger}_{\{n\}}(in)$, $b^{\dagger}_{\{n\}}(in)$, $a_{\{n\}}(in)$, $b_{\{n\}}(in)$ are creation and annihilation operators of in-particles and antiparticles respectively and $a^{\dagger}_{\{n\}}(out)$,$b^{\dagger}_{\{n\}}(out)$, $a_{\{n\}}(out)$, $b_{\{n\}}(out)$ are ones of out-particles and antiparticles, $\{n\}$ are possible quantum numbers. For example, let us calculate the mean numbers of antiparticles created (which are also equal to the numbers of pairs created) by the external field from the in-vacuum $|0, in>$ with a given quantum number p_D, n, r. By using relations (81) and (21) one finds representation of this quantity:

$$N_{p_D,n,r} =< 0, in|b^{\dagger}_{p_D,n,r}(out)b_{p_D,n,r}(out)|0, in >=$$
$$\left(G\left({}_{+}|^{-}\right)^{\dagger} G\left({}_{+}|^{-}\right)\right)_{p_D,n,r,p'_D,n',r}\Big|_{p_D=p'_D,\ n=n'}, \tag{82}$$
$$\left(G\left({}_{+}|^{-}\right)^{\dagger} G\left({}_{+}|^{-}\right)\right)_{p_D,n,r,p'_D,n',r} =$$
$$\int_{-\infty}^{+\infty} \left(G\left({}_{+}|^{-}\right) G\left({}_{+}|^{-}\right)^{\dagger}\right)_{p_-,n,r,p_-,n',r} M^*(p_D,p_-)M(p'_D,p_-)\frac{dp_-}{2\pi qE}. \tag{83}$$

where the standard space coordinate volume regularization was used, so that $\delta(p_j - p'_j) \to \delta_{p_j,p'_j}$. Then, by using formulas (28) and (29) one gets

$$N_{p_D,n,r} = \int_{-\infty}^{+\infty} {}^{-}\mathcal{D}_{nn} M^*(p_D,p_-)M(p'_D,p_-)\frac{dp_-}{2\pi qE}\Big|_{p_D=p'_D}. \tag{84}$$

If L_D is the length of the correspondent edge of the space box then the maximum wave length of the plane wave is $2L_D$. Thus, using the Fourier-series expansion of ${}^{-}\mathcal{D}$ one gets

$$N_{p_D,n,r} = \frac{1}{L_D}\int_0^{L_D} {}^{-}\mathcal{D}_{nn} dp_-. \tag{85}$$

One can calculate the quantities ${}^{-}_{+}\mathcal{D}_{nn'}$, given by Eq. (29), taking into account that the operator $\pi_\perp$ is Hermitian, and

$$e^{iK^*\pi_\perp}e^{-iK\pi_\perp} = \exp\left\{-\frac{i}{2}K^*qFK - i\left(K - K^*\right)\pi_\perp\right\}.$$

Then all the integrals over x^j in form (29) can be expressed in terms of the Laguerre polynomials [25]. In the special case when $n = n'$ one has

$${}^{-}\mathcal{D}_{nn} = \exp\left\{-\pi\lambda - \mathrm{Im}\ {}^{-}K(x_-)\left(qf(p_-/qE) + qF\mathrm{Re}\ {}^{-}K(x_-)\right) - \sum_{j=1}^{[d/2]-1} h_j\right\}\prod_{j=1}^{[d/2]-1} L_{n_j}(2h_j), \tag{86}$$
$$h_j = -|qH_j|((\mathrm{Im}\ {}^{-}K_{2j-1}(x_-))^2 + (\mathrm{Im}\ {}^{-}K_{2j}(x_-))^2),\quad \pi_- > 0.$$

Remember that we are discussing the case in which the constant electric field acts for an infinite time. However, one can analyze the problem in finite times $T = x^0_{out} - x^0_{in}$, acting similar to [13]. Then the mean numbers of pairs created by the external field $N_{p_D,n,r}$ are the same if time T is large enough: $\sqrt{qE}\,T \gg 1$, $\sqrt{qE}T \gg \lambda$ and $qET \gg |p_D|$.

Summing over the quantum numbers in (85), one can find the total number of pairs created from the vacuum. Using standard regularization with respect to the $(d-1)$-dimensional spatial volume $V_{(d-1)}$ and special regularization with respect to time T of acting of a constant electric field [13], where $\int dp_D N_{p_D,n,r} = qETN_{p_D,n,r}$, one gets

$$
\begin{aligned}
&N = V_{(d-1)} n^{cr} , \\
&n^{cr} = J_{(d)} \frac{Tm^2 \beta(1)}{2^{(d-1)} \pi^{d/2}} \frac{E}{E_c} \exp\left\{ -\pi \frac{E_c}{E} \right\} ,
\end{aligned} \tag{87}
$$

where n^{cr} is the number density of the created pairs for time T, and the coefficient $\beta(1)$ is defined by the next formula as a special case on $l = 1$:

$$
\begin{aligned}
&\beta(l) = \prod_{j=1}^{(d-2)/2} \{ qH_j \coth(l\pi H_j/E) \} \ , \ \ d \text{ is even} \ , \\
&\beta(l) = \left(\frac{m^2 E}{n\pi E_c} \right)^{\frac{1}{2}} \prod_{j=1}^{(d-3)/2} \{ qH_j \coth(l\pi H_j/E) \} \ , \ \ d \text{ is odd} \ .
\end{aligned} \tag{88}
$$

Here, $E_c = m^2/|q|$ is the characteristic value of a constant electric field strength. This quantity N does not depend on the parameters of plane-wave field and is the same as the number of pairs created in a constant and uniform field [13]. The corresponding formulas for the $d = 4$ case were first written in [26], and, in fact, can be derived easily from the Schwinger formulas [22].

By using proper-time kernel $f(x, x', s)$ (73) one can construct the d dimensional form of the Schwinger out-in effective action [22]

$$
\Gamma_{out-in} = \frac{1}{2} \text{tr} \left\{ \int dx \int_0^\infty s^{-1} f(x, x, s) \, ds \right\} \ . \tag{89}
$$

It is not dependent on the parameters of the plane-wave field. And it is not amazing because the effective action is a function of the field invariants only which do not depend on the plane wave. Taking into account formulas (65) one can find the trace in (89) representation

$$
\rho(s) = \text{tr} \left\{ \exp\left(-i \frac{q}{2} \sigma^{\mu\nu} F_{\mu\nu} s \right) \right\} = 2^{[d/2]} \cosh(qEs) \prod_{j=1}^{[(d-2)/2]} \cos(qH_j s). \tag{90}
$$

Then, one calculates the probability for a vacuum to remain a vacuum by using Schwinger method [22],

$$
\begin{aligned}
&P_v = \exp\left(-2 \text{Im} \Gamma_{out-in} \right) = \exp\left\{ -\mu N \right\} , \\
&\mu = \sum_{l=0}^{\infty} \frac{\beta(l+1)}{(l+1)\beta(1)} \exp\left\{ -l\pi \frac{E_c}{E} \right\} \ ,
\end{aligned} \tag{91}
$$

where $\beta(l)$ is defined by (88). This result coincides with the result from [13], and in $d = 4$ the results coincide with Refs. [22,26].

Let the operator of the current of the Dirac field operator $\psi(x)$ have the form

$$j_\mu = \frac{q}{2}\left[\bar{\psi}(x), \gamma_\mu\, \psi(x)\right], \tag{92}$$

and operator of metric energy-momentum tensor (EMT) of the Dirac field operator has the form

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2}\left(T^{can}_{\mu\nu} + T^{can}_{\nu\mu}\right), \, , \\ T^{can}_{\mu\nu} &= \frac{1}{4}\left\{\left[\bar{\psi}(x), \gamma_\mu \mathcal{P}_\nu\, \psi(x)\right] + \left[\mathcal{P}^*_\nu \bar{\psi}(x), \gamma_\mu\, \psi(x)\right]\right\} \, , \end{aligned} \tag{93}$$

where $T^{can}_{\mu\nu}$ is the canonical EMT operator. We are going to discuss the following matrix elements with these operators:

$$< j_\mu >^c = < 0, out|j_\mu|0, in > c_v^{-1} \, , \tag{94}$$

$$< T_{\mu\nu} >^c = < 0, out|T_{\mu\nu}|0, in > c_v^{-1} \, , \tag{95}$$

$$< j_\mu >^{in} = < 0, in|j_\mu|0, in > \, , \tag{96}$$

$$< T_{\mu\nu} >^{in} = < 0, in|T_{\mu\nu}|0, in > \, , \tag{97}$$

$$< j_\mu >^{out} = < 0, out|j_\mu|0, out > \, , \tag{98}$$

$$< T_{\mu\nu} >^{out} = < 0, out|T_{\mu\nu}|0, out > \quad . \tag{99}$$

Using the Green functions which were found before, one can present these matrix elements in the following form:

$$< j_\mu >^c = iq\mathrm{tr}\left\{\gamma_\mu S^c(x,x)\right\} = iq\ \mathrm{tr}\left\{\gamma_\mu \gamma^\kappa \mathcal{P}_\kappa \Delta^c(x,x')\right\}\big|_{x=x'} \, , \tag{100}$$

$$\begin{aligned} < T_{\mu\nu} >^c &= i/4\ \mathrm{tr}\left\{\left(\gamma_\mu\left(\mathcal{P}_\nu + \mathcal{P}'^*_\nu\right) + \gamma_\nu\left(\mathcal{P}_\mu + \mathcal{P}'^*_\mu\right)\right) S^c(x,x')\right\}\Big|_{x=x'} \\ &= i\ \mathrm{tr}\left\{B_{\mu\nu}\Delta^c(x,x')\right\}\big|_{x=x'} \, , \end{aligned} \tag{101}$$

$$< j_\mu >^{in} = < j_\mu >^c - < j_\mu >^a \, , \tag{102}$$

$$< j_\mu >^{out} = < j_\mu >^c - < j_\mu >^p \, , \tag{103}$$

$$< T_{\mu\nu} >^{in} = < T_{\mu\nu} >^{c} - < T_{\mu\nu} >^{a} , \quad (104)$$

$$< T_{\mu\nu} >^{out} = < T_{\mu\nu} >^{c} - < T_{\mu\nu} >^{p} , \quad (105)$$

$$< j_{\mu} >^{(a,p)} = iq \, \mathrm{tr} \left\{ \gamma_{\mu} \gamma^{\kappa} \mathcal{P}_{\kappa} \Delta^{a,p}(x, x') \right\} \big|_{x=x'} \; , \quad (106)$$

$$< T_{\mu\nu} >^{(a,p)} = i \, \mathrm{tr} \left\{ B_{\mu\nu} \Delta^{a,p}(x, x') \right\} \big|_{x=x'} \; , \quad (107)$$

$$B_{\mu\nu} = 1/4 \left\{ \gamma_{\mu} \left(\mathcal{P}_{\nu} + \mathcal{P'}^{*}_{\nu} \right) + \gamma_{\nu} \left(\mathcal{P}_{\mu} + \mathcal{P'}^{*}_{\mu} \right) \right\} \gamma^{\kappa} \mathcal{P}_{\kappa} \, ,$$

where the Green functions are given by Eqs. (76), (77) and the relation

$$\Delta^{c}(x, x) = \frac{1}{2} \left[\Delta^{-}(x, x) - \Delta^{+}(x, x) \right]$$

is used. It is convenient to represent $< j_{\mu} >^{a,p}$ and $< T_{\mu\nu} >^{a,p}$ as follows:

$$- < j_{\mu} >^{a} = < j_{\mu} >^{(1)} + < j_{\mu} >^{(2)} , \quad (108)$$

$$- < j_{\mu} >^{p} = < j_{\mu} >^{(1)} - < j_{\mu} >^{(2)} , \quad (109)$$

$$- < T_{\mu\nu} >^{a} = < T_{\mu\nu} >^{(1)} + < T_{\mu\nu} >^{(2)} , \quad (110)$$

$$- < T_{\mu\nu} >^{p} = < T_{\mu\nu} >^{(1)} - < T_{\mu\nu} >^{(2)} , \quad (111)$$

where

$$< j_{\mu} >^{(1)} = iq \, \mathrm{tr} \left\{ \gamma_{\mu} \gamma^{\kappa} \mathcal{P}_{\kappa} \Delta^{(1)}(x, x') \right\} \Big|_{x=x'} \; , \quad (112)$$

$$< T_{\mu\nu} >^{(1)} = i \, \mathrm{tr} \left\{ B_{\mu\nu} \Delta^{(1)}(x, x') \right\} \Big|_{x=x'} \; , \quad (113)$$

$$\Delta^{(1)}(x, x') = -\frac{1}{2} \int_{\Gamma_3 + \Gamma_2 + \Gamma_a} f(x, x', s) ds \quad , \quad (114)$$

and all contributions with derivatives of $\Theta(\pm y^{D})$ functions, which are formally divergent, are included in terms $< j_{\mu} >^{(2)}$ and $< T_{\mu\nu} >^{(2)}$. The nature of such divergences is connected with infinite time T of acting of a constant electric field and was discussed in [13] (see also below).

The components $< j_{\mu} >^{(1,2)}$ and $< T_{\mu\nu} >^{(1,2)}$ can not be calculated in the framework of the perturbation theory with respect to the external background or in the framework of the WKB method. Among them only the term $< j_{\mu} >^{in}$ was calculated before in $d = 4$ [3]. Only expression (101) for $< T_{\mu\nu} >^{c}$ has to be regularized and renormalized because of the ultraviolet divergences. Expression (100) for the term $< j_\mu >^c$ is finite after the regularization lifting and equal to zero. The terms $< j_\mu >^{(1)}$ and $< T_{\mu\nu} >^{(1)}$ are also finite, and $< j_\mu >^{(1)} = 0$. The terms $< j_\mu >^{(2)}$ and $< T_{\mu\nu} >^{(2)}$ have to be regularized with respect to time T of acting of a constant electric field [13] and do not have standard ultraviolet divergences. That is consistent with the fact that the ultraviolet divergences have a local nature and result (as in the theory without external field) from the leading local terms at $s \to +0$. The nonzero contributions to the expressions $< j_\mu >^{(2)}$ and $< T_{\mu\nu} >^{(1,2)}$ are related to global features of the theory and indicate the vacuum instability.

At asymptotic region $x_0 = T/2 \to +\infty$ the densities of current and the EMT of particles created are

$$j_\mu^{cr} = \frac{\int d\mathbf{x} \left(< j_\mu >^{in} - < j_\mu >^{out}\right)}{\int d\mathbf{x}} \,, \tag{115}$$

$$T_{\mu\nu}^{cr} = \frac{\int d\mathbf{x} \left(< T_{\mu\nu} >^{in} - < T_{\mu\nu} >^{out}\right)}{\int d\mathbf{x}} \,, \tag{116}$$

according to definitions (96) - (99).

Then, using representations (102) - (107), and taking into account terms $< j_\mu >^{(2)}$ and $< T_{\mu\nu} >^{(2)}$ which are uniform, one gets from (115) and (116),

$$j_\mu^{cr} = 2 < j_\mu >^{(2)} \,, \tag{117}$$

$$T_{\mu\nu}^{cr} = 2 < T_{\mu\nu} >^{(2)} \quad . \tag{118}$$

Using special regularization with respect to time T of acting of a constant electric field [13] we can interpret these divergent terms and, correct to first order of $\sqrt{qE}T$, obtain

$$< j_\mu >^{cr} = 2|q| n^{cr} \delta_\mu^D , \tag{119}$$

$$< T_{00} >^{cr} = < T_{DD} >^{cr} = qETn^{cr} \; . \tag{120}$$

Other components of $< T_{\mu\nu} >^{cr}$ are of the order of $\ln(\sqrt{qE}T)$.

To study the backreaction of particles created on the electromagnetic field and metrics one needs the expressions $< j_\mu >^{in}$ and $< T_{\mu\nu} >^{in}$. We found the form of $< j_\mu >^{in} = < j_\mu >^{(2)} = 1/2 < j_\mu >^{cr}$ and $< T_{\mu\nu} >^{(2)} = 1/2 < T_{\mu\nu} >^{cr}$. Let us calculate other terms in expressions (104). By using formulas (65) one can find traces in (101) and (107) representations. One has the next nonzero traces: one presented by expression (90) and

$$
\begin{aligned}
&\operatorname{tr}\left\{\gamma^0\gamma^D \exp\left(-i\frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}s\right)\right\} = \tanh(qEs)\rho(s)\ , \\
&\operatorname{tr}\left\{\gamma^{2j}\gamma^{2j-1} \exp\left(-i\frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}s\right)\right\} = -\tan(qH_j s)\rho(s)\ .
\end{aligned} \tag{121}
$$

Then, nonzero components of $< T_{\mu\nu} >^c$ and $< T_{\mu\nu} >^{(1)}$ are

$$
< T_{\mu\nu} >^c = \int_{\Gamma_c} \tau_{\mu\nu}(s)ds\ , \tag{122}
$$

$$
< T_{\mu\nu} >^{(1)} = -\frac{1}{2}\int_{\Gamma_3+\Gamma_2+\Gamma_a} \tau_{\mu\nu}(s)ds\ , \tag{123}
$$

$$
\begin{aligned}
&\tau_{\mu\nu}(s) = b_\mu(s)\rho(s)f^{(0)}(x,x,s) \text{ if } \mu=\nu\ , \\
&b_D(s) = -b_0(s) = \frac{qE}{\sinh(2qEs)},\ \ b_{2j}(s) = b_{2j-1}(s) = \frac{qH_j}{\sin(2qH_j s)}\ .
\end{aligned}
$$

V. ACKNOWLEDGMENTS

D.M.G. thanks the Brazilian foundations CNPq for support. S.P.G. thanks the Brazilian foundation CAPES for support, and the Department of Physics of UEL and Department of Mathematical Physics of USP for their hospitality.

[1] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin 1985).

[2] A.A. Grib, S.G. Mamaev, and V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Atomizdat, Moscow 1988; Friedmann Laboratory Publishing, St. Petersburg 1994).

[3] E.S. Fradkin, D.M. Gitman and S.M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin 1991).

[4] W.H. Furry, Phys. Rev. **81**, 115 (1951).

[5] D.M. Gitman, J. Phys. A **10**, 2007 (1977); D.M. Gitman and S.P. Gavrilov, Sov. Phys. Journ. No.1, 94 (1977); S.P. Gavrilov and D.M. Gitman, Sov. Phys. Journ. No.6, 491 (1980).

[6] V.G. Bagrov and D.M. Gitman, *Exact Solutions of Relativistic Wave Equations* (Kluwer Academic, Amsterdam 1990).

[7] V.G. Bagrov, D.M. Gitman, and A.V. Jushin, Phys. Rev. D**12**, 3200 (1975); V.G. Bagrov, D.M. Gitman, and V.N. Shapovalov, J. Math. Phys. **23**, 2558 (1982).

[8] S.P. Gavrilov, D.M. Gitman and Sh.M. Shvartsman, Sov. J. Nucl. Phys. (USA) **29**, 567 (1979); 715 (1979); Kratkie Soobshenia po Fizike (P.N. Lebedev Inst.) No 2, 22 (1979).

[9] S.P. Gavrilov and D.M. Gitman, Sov. Phys. Journ. No.4, 364 (1983); S.P. Gavrilov and D.M. Gitman, Sov. Phys. Journ. No. 9, 775(1982); S.P. Gavrilov and D.M. Gitman, Izw. VUZov Fizika (Sov. Phys. Journ.) No. 5, 108 (1981); Yu.Yu. Volfengaut, S.P. Gavrilov, D.M. Gitman and Sh.M. Shvartsman, Sov. Journ. Nucl. Phys. **33**, 386(1981); D.M. Gitman, M.D. Noskov and Sh.M. Shvartsman, Int. J. Mod. Phys. A**6**, 4437 (1991).

[10] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol and Philadelphia 1992)

[11] G. Schäfer and H. Dehnen, J. Phys. A **13**, 517 (1980); I.L. Buchbinder and S.D. Odintsov, Izv. VUZov Fizika (Sov. Phys. Journ.) No.5, 12 (1982).

[12] S.P. Gavrilov, D.M. Gitman and S.D. Odintsov, Int. J. Mod. Phys. A **12**, 4837 (1997).

[13] S.P. Gavrilov and D.M. Gitman, Phys. Rev. D **53**, 7162 (1996).

[14] R. Brauer and H. Weyl, Amer. J. Math. **57**, 425 (1935).

[15] V.G. Bagrov, D.M. Gitman and V.A. Kuchin, in *Actual Problems of Theoretical Physics* (MGU, Moscow 1976) 334: S.P. Gavrilov and D.M. Gitman, Sov. J. Nucl. Phys. (USA) **51**, 1040 (1990).

[16] P.F. Redmond, J. Math. Phys. **6**, 1163 (1965).

[17] D.M. Wolkov, Zeit. Phys. **94**, 250 (1935).

[18] N.B. Narozhny and A.I. Nikishov, Theor. Math. Phys. **26**, 9 (1976); N.B. Narozhny and A.I. Nikishov, in *Problems of Intense Field Quantum Electrodynamics*, Proc. P.N. Lebedev Phys. Inst. **168**, 175 (Nauka, Moscow 1986)

[19] *Tables of Integral Transformations* (Bateman Manuscript Project), edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1954), Vol. 2.

[20] *Higher Transcendental functions* (Bateman Manuscript Project), edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1953), Vol. 2.

[21] E.S. Fradkin and D.M. Gitman, Preprint MIT (1978) pp. 1-58; Preprint KFKI-1979-83, pp. 1-105; Preprint PhIAN (P.N. Lebedev Institute) 106 (1979) pp. 1-62; 107 (1979) 1-40; Fortschr. Phys. **29**, 381 (1981).

[22] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[23] V. Fock, Phys. Z. Sowjetunion **12**, 404 (1937).

[24] S.P. Gavrilov and D.M. Gitman, J. Math. Phys. **37**, 3118 (1996).

[25] I.S. Gradshtein and I.M. Rizhik *Tables of Integrals, Sums, Series and Productions* (Nauka,

Moscow, 1971).

[26] A.I. Nikishov, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969) [Sov. Phys. JETP **30**, 660 (1970)]; in *Quantum Electrodynamics of Phenomena in Intense Fields*, Proc. P.N. Lebedev Phys. Inst. **111**, 153 (Nauka, Moscow, 1979)